Theorem. If $x(t)$ is real and even, then its Fourier transform

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt$$

is real and even.

Proof.

Start with the definition of the Fourier transform:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt.$$

Using Euler's identity,

$$e^{-j\omega t} = \cos(\omega t) - j\sin(\omega t),$$

substitute to obtain

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\cos(\omega t)\, dt - j\int_{-\infty}^{\infty} x(t)\sin(\omega t)\, dt.$$

Since $x(t)$ is even, $\cos(\omega t)$ is even, and $\sin(\omega t)$ is odd, we have

$$x(t)\cos(\omega t) \text{ is even,} \qquad x(t)\sin(\omega t) \text{ is odd.}$$

The integral of an odd function over $\mathbb{R}$ is zero, so

$$\int_{-\infty}^{\infty} x(t)\sin(\omega t)\, dt = 0.$$

Thus,

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\cos(\omega t)\, dt,$$

which is purely real.

To show evenness, evaluate $X(-\omega)$:

$$X(-\omega) = \int_{-\infty}^{\infty} x(t)e^{j\omega t}\, dt = \int_{-\infty}^{\infty} x(t)\cos(\omega t)\, dt = X(\omega),$$

using the fact that $\cos(\omega t)$ is even.

Thus $X(\omega)$ is both real and even.

$\boxed{\text{If } x(t) \text{ is real and even, then } X(\omega) \text{ is real and even.}}$